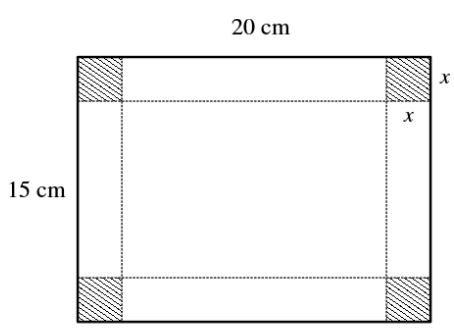
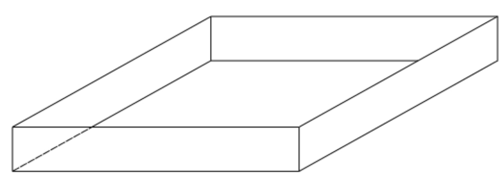
Solving Polynomial Equations

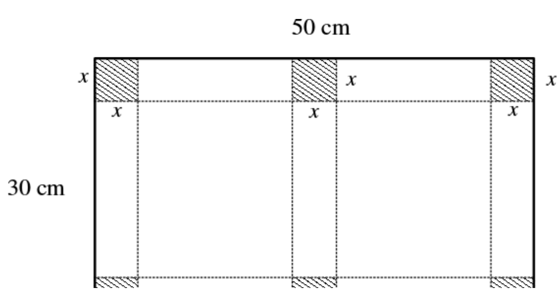
Exercises

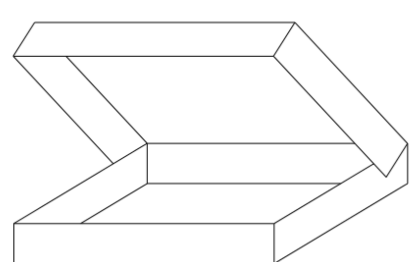
1. Solve the following polynomial equations by factoring where $x \in \mathbb{R}$.
   a. $x^3 - 5x^2 - 4x + 20 = 0$
   b. $2x^3 + 3x^2 = 11x + 6$
   c. $4x^2 = x^3 + 2x + 3$
   d. $x^4 - 7x^2 + 12 = 0$
   e. $2x^3 + 15 = 6x^2 + 5x$
   f. $2x(x^3 + 1) = x^2(4x + 1)$
   g. $2x^4 + 8x + 12 = 3x^2(x + 3)$
2. Find all possible roots of the polynomial equation where $x \in \mathbb{C}$.
   a. $2x^3 + 5x^2 + 14x + 6 = 0$
   b. $8x^4 = x$
   c. $x^2(4x^2 + 17) = 15$
3. If one root in each of the given equations is $x = 2$, determine the other roots. In the following equations, $k \in \mathbb{R}$.
   a. $3x^3 - 15x^2 + kx - 4 = 0$
   b. $25x^4 + kx^2 + 16 = 0$
4. Sketch a possible graph for each polynomial function, using the intercepts and end behaviour of the function.
   a. $y = 2x^3 - 12x^2 + 18x$
   b. $y = -x^3 + 4x^2 + x - 4$
   c. $y = x^4 - 8x^2 + 16$
5. Explain why
   a. $15x^5 + 4x^4 + 9x^2 + 7x + 380 = 0$ has at least one real root.
   b. $5x^6 + 3x^4 + 8x^2 + 120 = 0$ has no real roots.
6. A rectangular holding tank is x metres deep, $(6x - 8)$ metres long, and $(6x - 16)$ metres wide. Find the dimensions of the tank with a volume of $512\ \text{m}^3$.
7. The product of the squares of two consecutive integers is 1764. Find all possible values for the integers.
8. A rectangular sheet of metal with dimensions 20 cm by 15 cm is to be used to create an open top box by cutting a square, x cm by x cm, from each corner and bending up the sides. If a volume of $375\ \text{cm}^3$ is required, determine the side length of the squares that must be cut.

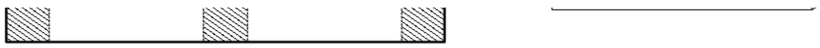




9. A box with a lid is to be created from a 50 cm by 30 cm piece of cardboard by cutting x by x squares from the four corners of the cardboard, and at the centre of the two sides, as shown in the diagram. Determine the function that represents the volume of the box in terms of x, and state the restrictions on x. If the box is to have a volume of $1750\ \text{cm}^3$, determine the side length of the squares that need to be cut

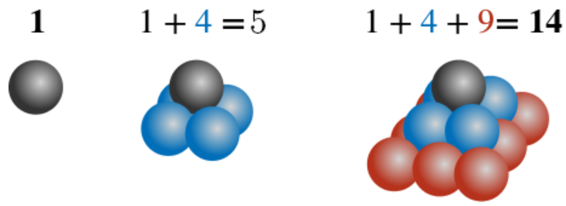






10. Solve $x^2(x^2+6)=5x^3-x+1, x \in \mathbb{R}$.

11. The first three square pyramidal numbers are $1, 5,$ and 14 as shown in the diagram. The number of balls in each layer of the pyramids is a perfect square.



The only pyramidal number, other than 1, that is a perfect square is 4900. How many layers are in the pyramid which contains 4900 balls?

Note: The sum of the first n perfect squares is $\frac{n(n+1)(2n+1)}{6}$.

12. Determine all possible values of $n, n \in \mathbb{R}$, such that the equation $3x^3+11x^2+8x+n=0$ has two equal real roots.

Solving Polynomial Equations

Partial Solutions

1. There is no solution provided for this question.

2. a. Let $f(x) = 2x^3 + 5x^2 + 14x + 6$. $f\left(-\frac{1}{2}\right) = 0$, so $(2x+1)$ is a factor of $f(x)$. Dividing gives

$$2x^3 + 5x^2 + 14x + 6 = 0$$
$$(2x+1)(x^2 + 2x + 6) = 0$$

so $2x + 1 = 0$ or $x^2 + 2x + 6 = 0$.

$$\begin{aligned} 2x + 1 &= 0 \\ x &= -\frac{1}{2} \end{aligned} \quad \text{or} \quad \begin{aligned} x &= \frac{-2 \pm \sqrt{4 - 4(6)}}{2} \\ &= \frac{-2 \pm \sqrt{-20}}{2} \\ &= \frac{-2 \pm i\sqrt{20}}{2} \\ &= \frac{-2 \pm i(2\sqrt{5})}{2} \\ &= -1 \pm i\sqrt{5} \end{aligned}$$

Therefore $x = -\frac{1}{2}, -1 \pm i\sqrt{5}$.

b. Let $f(x) = 8x^4 - x$. Factoring,

$$8x^4 - x = 0$$
$$x(8x^3 - 1) = 0$$
$$x(2x-1)(4x^2 + 2x + 1) = 0$$

$$x = 0 \quad \text{or} \quad \begin{aligned} 2x - 1 &= 0 \\ x &= \frac{1}{2} \end{aligned} \quad \text{or} \quad \begin{aligned} x &= \frac{-2 \pm \sqrt{4 - 4(4)}}{8} \\ &= \frac{-2 \pm \sqrt{-12}}{8} \\ &= \frac{-2 \pm i\sqrt{12}}{8} \\ &= \frac{-2 \pm i(2\sqrt{3})}{8} \\ &= \frac{-1 \pm i\sqrt{3}}{4} \end{aligned}$$

Therefore $x = 0, \frac{1}{2}, \dfrac{-1 \pm i\sqrt{3}}{4}$.

c. Rearranging the equation and factoring,

$$4x^4 + 17x^2 - 15 = 0$$
$$(4x^2 - 3)(x^2 + 5) = 0$$

$$\begin{aligned} 4x^2 - 3 &= 0 \\ x^2 &= \frac{3}{4} \\ x &= \pm\frac{\sqrt{3}}{2} \end{aligned} \quad \text{or} \quad \begin{aligned} x^2 + 5 &= 0 \\ x^2 &= -5 \\ x &= \pm i\sqrt{5} \end{aligned}$$

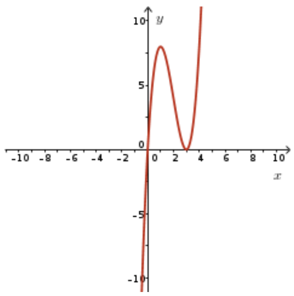
Therefore $x = \pm i\sqrt{5}, \ \pm\frac{\sqrt{3}}{2}$.

3. There is no solution provided for this question.

4. a. Factoring the equation,

$$\begin{aligned} y &= 2x^3 - 12x^2 + 18x \\ y &= 2x(x^2 - 6x + 9) \\ &= 2x(x-3)^2 \end{aligned}$$

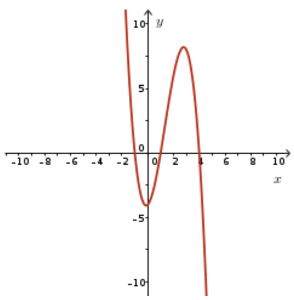
Setting $y = 0$, we see that the function has x-intercepts at $x = 0$ and $x = 3$ (order 2). The y-intercept is 0. The function is cubic with positive leading coefficient, so $y \to \infty$ as $x \to \infty$ and $y \to -\infty$ as $x \to -\infty$. A possible sketch is



b. Factoring the equation by grouping,

$$\begin{aligned} y &= -x^3 + 4x^2 + x - 4 \\ y &= -x^2(x-4) + (x-4) \\ &= (x-4)(-x^2+1) \\ &= -(x-4)(x-1)(x+1) \end{aligned}$$

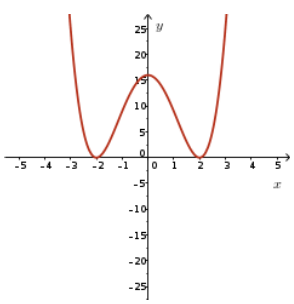
Setting $y = 0$, we see that the function has x-intercepts at $x = -1, 1$ and 4. The y-intercept is -4. The function is cubic with negative leading coefficient, so $y \to -\infty$ as $x \to \infty$ and $y \to \infty$ as $x \to -\infty$. A possible sketch is



c. Factoring the equation,

$$\begin{aligned} y &= x^4 - 8x^2 + 16 \\ y &= (x^2-4)^2 \\ &= \left[(x-2)(x+2)\right]^2 \\ &= (x-2)^2(x+2)^2 \end{aligned}$$

Setting $y = 0$, we see that the function has x-intercepts at $x = \pm 2$ (both of order 2). The y-intercept is 16. The function is quartic with positive leading coefficient, so $y \to \infty$ as $x \to \pm\infty$. A possible sketch is



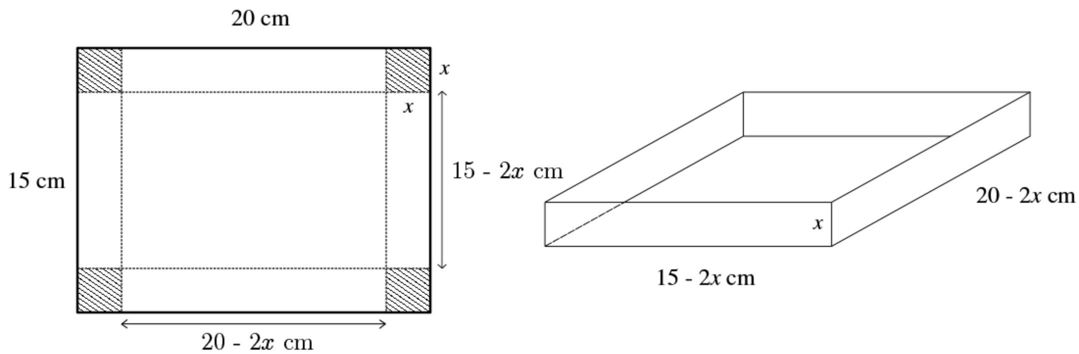
5. There is no solution provided for this question.
6. Using the formula for volume of a rectangular solid, $V(x) = lwh$, we have $V(x) = x(6x-8)(6x-16)$.
The height, $h = x$, is a positive quantity so $x > 0$.
The length, $l = 6x - 8$, must be positive so $6x - 8 > 0$ and $x > \frac{4}{3}$.
The width, $w = 6x - 16$, is positive so $6x - 16 > 0$ and $x > \frac{8}{3}$.
Thus $x > \frac{8}{3}, x \in \mathbb{R}$.
Since the volume is 512,

$$\begin{aligned} 512 &= x(6x-8)(6x-16) \\ 512 &= (6x^2-8x)(6x-16) \\ 512 &= 36x^3 - 144x^2 + 128x \\ 0 &= 36x^3 - 144x^2 + 128x - 512 \\ 0 &= 9x^3 - 36x^2 + 32x - 128 \\ 0 &= 9x^2(x-4) + 32(x-4) \\ 0 &= (x-4)(9x^2+32) \end{aligned}$$

so $x-4=0$ or $9x^2+32=0$. But $9x^2+32=0$ has no real roots, so $x=4>\frac{8}{3}$ is the only solution. Therefore the height of the tank is $h=x=4$ metres, the width is $w=6(4)-16=8$ metres, and the length is $l=6(4)-8=16$ metres.

7. There is no solution provided for this question.

8. Let x represent the side length, in centimetres, of the cut out squares. Therefore the length of the base is $20-2x$ and the width is $15-2x$ (as shown).



The height, $h=x$, is positive so $x>0$.
The length of the box's base, $l=20-2x$, is positive so $x<10$.
The width, $w=15-2x$, is positive so $x<\frac{15}{2}$. Thus $0<x<\frac{15}{2}, x\in\mathbb{R}$. Since the volume is 375,

$$\begin{aligned} V(x) &= hlw \\ &= x(20-2x)(15-2x) \\ 375 &= x(20-2x)(15-2x) \\ &= (20x-2x^2)(15-2x) \\ &= 4x^3-70x^2+300x \\ 0 &= 4x^3-70x^2+300x-375 \end{aligned}$$

Let $P(x)=4x^3-70x^2+300x-375$. Using the rational roots theorem to generate test values, while considering the restrictions on x, gives

$$\{\pm1,\pm3,\pm5,\pm\frac{1}{2},\pm\frac{3}{2},\pm\frac{5}{2},\pm\frac{15}{2},\pm\frac{1}{4},\pm\frac{3}{4},\pm\frac{5}{4},\pm\frac{15}{4},\pm\frac{25}{4}\}$$

as possible roots to the equation $P(x)=0$.
Observe that $P\left(\frac{5}{2}\right)=0$, so $\frac{5}{2}$ is a zero and $2x-5$ is a factor. Factoring gives

$$\begin{aligned} 4x^3-70x^2+300x-375 &= 0 \\ (2x-5)(2x^2-30x+75) &= 0 \end{aligned}$$

so $2x-5=0$ and $x=\frac{5}{2}$, or $2x^2-30x+75=0$ and $x=\dfrac{15\pm5\sqrt{3}}{2}$.

Thus $x=\frac{5}{2}, \dfrac{15\pm5\sqrt{3}}{2}$ are the roots of $P(x)=0$; applying the physical restriction $0<x<\frac{15}{2}$, we see that there are two possible solutions,

$$\begin{aligned} x &= \frac{5}{2} = 2.5 \text{ cm} \\ x &= \frac{15-5\sqrt{3}}{2} \approx 3.17 \text{ cm} \end{aligned}$$

Therefore, a square of side length 2.5 cm or 3.17 cm must be cut from each corner to produce a box with a volume of $375\ \text{m}^2$.

9. There is no solution provided for this question.

10.
$$\begin{aligned} x^2(x^2+6) &= 5x^3-x+1 \\ x^4+6x^2 &= 5x^3-x+1 \\ 0 &= -x^4+5x^3-6x^2-x+1 \\ &= x^4-5x^3+6x^2+x-1 \end{aligned}$$

Let $P(x)=x^4-5x^3+6x^2+x-1$. Using the rational roots theorem, we test if values $x\in\{\pm1\}$ satisfy $P(x)=0$. However, $P(\pm1)\neq0$, so there are no rational roots. $P(x)$ has no linear factors with integer or rational coefficients/terms.
Determining the two quadratic factors, (ax^2+bx+c) and (dx^2+ex+f),

$$\begin{aligned} P(x) &= x^4-5x^3+6x^2+x-1 \\ &= (ax^2+bx+c)(dx^2+ex+f) \\ &= (ad)x^4+(bd+ae)x^3+(cd+af+be)x^2+(bf+ce)x+cf \end{aligned}$$

Let $a=d=1$ and $c=1, d=-1$. Then

$$\begin{aligned} (x^2+bx+1)(x^2+ex-1) &= x^4-5x^3+6x^2+x-1 \\ x^4+(b+e)x^3+(be)x^2+(e-b)x-1 &= x^4-5x^3+6x^2+x-1 \end{aligned}$$

Equating coefficients,

$$\begin{aligned} b+e &= -5 && (1) \\ be &= 6 && (2) \\ e-b &= 1 && (3) \end{aligned}$$

Adding (1) to (3), $2e=-4$ and so $e=-2$.
Substituting $e=-2$ into (1), $-2+b=-5$ so $b=-3$.
Checking with (2), $be=(-2)(-3)=6$; thus

$$P(x) = x^4 - 5x^3 + 6x^2 + x - 1 = (x^2 - 2x - 1)(x^2 - 3x + 1)$$

Solving $P(x) = 0$,

$$\begin{aligned} x^2 - 2x - 1 &= 0 \\ x &= \frac{2 \pm \sqrt{8}}{2} \\ &= 1 \pm \sqrt{2} \end{aligned}$$

and

$$\begin{aligned} x^2 - 3x + 1 &= 0 \\ x &= \frac{3 \pm \sqrt{5}}{2} \end{aligned}$$

Therefore the solution to $x^2(x^2 + 6) = 5x^3 - x + 1, x \in \mathbb{R}$ is $x = 1 \pm \sqrt{2}, \frac{3 \pm \sqrt{5}}{2}$.

11. There is no solution provided for this question.

12. Let r represent the root of order 2. Then $3x^3 + 11x^2 + 8x + n = (x - r)(x - r)(ax + b), a, b \in \mathbb{R}$. Since a will be the coefficient of the x^3 term, $a = 3$, and

$$\begin{aligned} 3x^3 + 11x^2 + 8x + n &= (x^2 - 2rx + r^2)(3x + b) \\ &= 3x^3 + bx^2 - 6rx^2 - 2brx + 3r^2x + br^2 \\ &= 3x^3 + (b - 6r)x^2 + (3r^2 - 2br)x + br^2 \end{aligned}$$

Equating coefficients,

$$11 = b - 6r \quad (1)$$
$$8 = 3r^2 - 2br \quad (2)$$
$$r^2 b = n \quad (3)$$

From (1), $b = 11 + 6r$. Substituting (1) into (2),

$$\begin{aligned} 8 &= 3r^2 - 2(11 + 6r)r \\ 0 &= -9r^2 - 22r - 8 \\ 0 &= 9r^2 + 22r + 8 \\ &= (9r + 4)(r + 2) \\ r &= -\frac{4}{9}, -2 \quad (4) \end{aligned}$$

Substituting (1) into (3),

$$\begin{aligned} n &= r^2 b \\ &= r^2(11 + 6r) \\ &= 11r^2 + 6r^3 \quad (5) \end{aligned}$$

Substituting (4) into (5),

$$\begin{aligned} n &= 11\left(-\frac{4}{9}\right)^2 + 6\left(-\frac{4}{9}\right)^3 \\ &= \frac{176}{81} - \frac{384}{729} \\ &= \frac{1200}{729} \\ &= \frac{400}{243} \end{aligned}$$

and

$$\begin{aligned} n &= 11(-2)^2 + 6(-2)^3 \\ &= 44 - 48 \\ &= -4 \end{aligned}$$

Therefore the possible values for n are $n = \frac{400}{243}, -4$.